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Volume 4: The Simplest Mathematics

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Book 1: Logic and Mathematics (Unpublished Papers)

Paper 4: The Logic of QuantityP

§11. Measurement

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142. It was necessary to say something about imaginaries before coming to the subject of **measurement** since the modern theory of **measurement** (due to the researches of Cayley, Clifford, Klein, etc.) depends essentially upon imaginaries.

Let us first consider **measurement** in one dimension. There is a certain absurdity in talking about **measurement** in one dimension. This is seen in the instance of time. Suppose we only knew the flow of our inward sensations, but nothing spread into two dimensions, how could one interval of time be compared with another? Certainly, their contents might be so alike that we should judge them equivalent. But that is not shoving a scale along. It does not enable us to compare intervals unless they happen to have similar contents. However, it is convenient to put that consideration aside, and to begin (with Klein) at unidimensional **measurement** .

We are to measure, then, along a line. We will, for formal rhetoric's sake, conceive that line as returning into itself. We will, first, in order that we may apply numerical algebra, give a preliminary numbering to all the points of that line, so that every point has a number and but one number, and every real number, positive or negative, rational or surd, has a point and but one point, and so that the succession of any four numbers is the same as the succession on the line of the four corresponding points. Now, we must make a scale to shift along that line. We must imagine that we have a movable line which lies everywhere in coincidence with the fixed line, and which can be shifted. In the shifting, parts of it may become expanded or contracted, for we cannot tell whether they do or not unless we had some third standard to shift along to tell us; and then the same question would arise. But the continuity and succession of points shall not be broken in the shifting; and moreover, when the movable line has any one point brought back to coincidence with a former position, all the points shall be brought back. Now imagine all this extended to the imaginary numbers. Then, it is shown in the mathematical theory of functions, that if x be the number against which any point of the movable line falls in any one position and y be the number the same point falls against in any other position, it follows, because for each value of x there is just one value of y and for each value of y just one value of x, that x and y are connected by an equation linear in each, that is, an equation of the form

$$xy+Ax+By+C = 0.$$

This gives

$$y = -(Ax+C)/(x+B).$$

Now this is a function which forms the subject of some very beautiful and simple algebraical studies.†P1 It is convenient to put

$$A = B-\alpha-\beta$$
$$C = B^2-(\alpha+\beta)B+\alpha\beta.$$

Then

$$\begin{aligned} y &= ((\alpha+\beta-B)x-B^2+(\alpha+\beta)B-\alpha\beta)/(x+B) \\ &= (((\alpha+\beta)x+(\alpha+\beta)B-\alpha\beta)/(x+B))-B \\ &= (((\alpha^2-\beta^2)(x+B)-\alpha\beta(\alpha-\beta))/((x+B)(\alpha-B)))-B \\ &= (((x+B-\beta)\alpha^2-(x+B-\alpha)\beta^2))/((x+B-\beta)\alpha^1-(x+B-\alpha)\beta^1)))-B \end{aligned}$$

But

$$x = ((x+B-\beta)\alpha^1-(x+B-\alpha)\beta^1)/((x+B-\beta)\alpha^\circ-(x+B-\alpha)\beta^\circ)-B.$$

So that the effect of the shifting has been to raise the exponents of α and β by 1.

It is easily proved that the same operation, performed any number t times, gives

$$((x+B-\beta)\alpha^{t+1}-(x+B-\alpha)\beta^{t+1})/((x+B-\beta)\alpha^t-(x+B-\alpha)\beta^t)-B$$

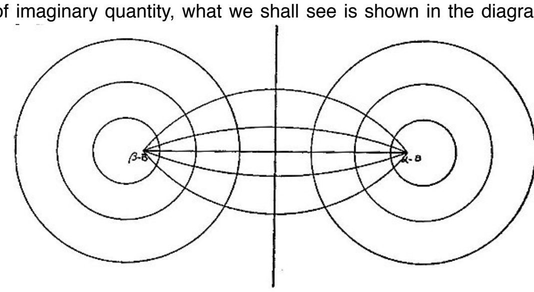
If α has a modulus greater than that of β, it is easily seen that when *t* becomes a very large positive number, the first terms of numerator and denominator will become indefinitely greater than the second terms and the value will indefinitely approximate to

$$\alpha\text{-}B.$$

But when *t* is a very large negative quantity, the reverse will occur, and the value will approximate toward

$$\beta\text{-}B.$$

143. If we look at the field of imaginary quantity, what we shall see is shown in the diagram.



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Here we have a stereographic projection of the globe. At the south pole is β-*B*; at the north pole, α-*B*. The parallels are not at equal intervals of latitude but are crowded together infinitely about the pole. Now an increase of *t* by unity carries a point of the scale along a meridian from one parallel to the one next nearer the north pole. But an addition to *t* of an *imaginary* quantity carries the point of the scale round along a parallel.

If the real line of the scale lies along a meridian all real shiftings of it will crowd its parts toward the north or the south pole; and the distance of either pole, as measured by the multitude of shiftings required to reach it, is *infinite*.†P1 The scale is *limited*, but *immeasurable*.

But if the real line of the scale lies along a parallel, real shiftings, that is shiftings from real points to real points, will carry it round, so that a finite number of shiftings will restore it to its first position. Such is the scale of rotatory displacement. It is *unlimited*, but finite, or *measurable*. A scale of **measurement**, in the sense here defined, cannot be both limited and finite. We seem to have such a scale in the **measurement** of probabilities. But it is not so. Absolute certainty, or probability 0 or probability 1 are unattainable; and therefore, the numbers attached to probabilities do not constitute any proper scale of **measurement**, *which can be shifted along*. But it is possible to construct a true scale for the **measurement** of belief.†P2 It was a part of the definition of a scale that in all its shiftings it should cover the whole of the line measured. ("For every point of the line a number of the scale in every position.") Hence the shifting can never be arrested by abuttal against a limit. If there is a limit, it must be at an immeasurable distance.

144. But there is a special case of **measurement**, very different from the one considered. Namely, it may happen that the nature of the shifting is such that [given] the equation

$$xy+Ax+By+C = 0,$$

where *A*, *B*, *C*, may have any values, real or imaginary, we have

$$C = 1/4(A+B)^2.$$

Substituting this in the expressions for *A* and *C* in terms of α, β, and *B*, we get

$$1/4(\alpha+\beta)^2 = \alpha\beta$$

or

$$\alpha = \beta.$$

This necessitates an altogether different treatment. In this case, we have

$$y = -((Ax+(1/4)(A+B)^2)/(x+B))$$
$$2y = -(A+B)+((B\text{-}A)(2x+A+B)/((B\text{-}A)+(2x+A+B)))$$
$$2x = -(A+B)+((B\text{-}A)(2x+A+B)/((B\text{-}A)+0(2x+A+B)))$$

And *t* shifts give

$$-(A+B)+((B\text{-}A)(2x+A+B)/((B\text{-}A)+t(2x+A+B))).$$

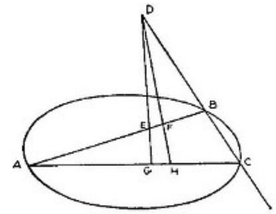
This gives for $t = \pm\infty$, $-(A+B)$. The scale is in this case then unlimited and immeasurable. This is the manner in which the Euclidean geometry virtually conceives lengths to be measured; but whether this method accords precisely with **measurement** by a rigid bar is a question to be decided experimentally, or irrationally, or not at all.

145. The fixed limits of **measurement** are very appropriately termed by mathematicians the Absolute.†P1 It is clear that even when **measurement** is not practical, even when we can hardly see how it ever can become so, the very idea of measuring a quantity, considerably illuminates our ideas about it. Naturally, the first question to be asked about a continuous quantity is whether the two points of its absolute coincide; if not, a second less important, but still significant question is whether they are in the real line of the scale or not. These are ultimately questions of fact which have to be decided by experimental indications; but the answers to them will have great bearing on philosophical and especially cosmogonical problems.

146. The mathematician does not by any means pretend that the above reasoning flawlessly establishes the absolute in every case. It is evident that it involves a premiss in regard to the imaginary points which only indirectly relates to anything in visible geometry, and which, of course, may be supposed not true. Nevertheless, the doctrine of the mathematical absolute holds with little doubt for all cases of **measurement**, because the assumptions virtually made will hardly ever fail.

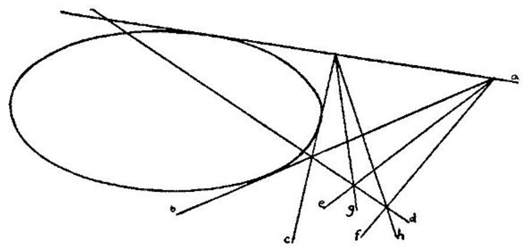
147. When we pass to **measurement** in several dimensions, there seems to be little difference between one number of dimensions and another; and therefore we may as well limit ourselves to studying **measurement** on a plane, the only spatial spread for which our intuition is altogether effortless.

Radiating from each point of the plane is a continuity of lines. Each of these has upon it its two absolute points (possibly imaginary, and even possibly coincident); and assuming these to be continuous, they form a curve which, being cut in two points only by any one line, is of the second order. That is, it is a conic section, though it may be an imaginary or even degenerate one.



Now as the foot has different lengths in different countries, so the ratios of units of lengths along different lines in the plane is somewhat arbitrary. But the **measurement** is so made that first, every point infinitely distant from another along a straight line is also infinitely distant along any broken line; and second, if two straight lines intersect at a point, *A*, on the absolute conic and respectively cut it again at *B* and *C*; and from *D*, any point collinear with *B* and *C*, two straight lines be drawn, the segments of the first two lines, *EF* and *GH*, which these cut off, are equal. I omit the geometrical proof that this involves no inconsistency. This proposition enables us to compare any two lengths.

148. We now have to consider angular magnitude. In the space of experience, the evidence is strong that, when we turn around and different landscapes pass panorama-wise before our vision we come round to the same direction, and not merely to a new world much like the old one. In fact, I know of no other theory for which the evidence is so strong as it is for this. But it is quite conceivable that this should not be so; there might be a world in which we never could get turned round but should always be turning to new objects. But certain conveniences result from assuming for the **measurement** of the angles between lines the same absolute conic which is assumed as the absolute of linear measure. Thus, it is assumed that two straight lines meeting at infinity have no inclination to one another, just as it is assumed that in a direction such that the opposite infinities should coincide, all other points would have no distance from one another. The latter is another way of saying that if a point is at an infinite distance from another point on a straight line, it is so on a broken line. The other assertion is that if an infinite turning is requisite to reach a line from one centre, it is equally so if you attempt to reach it by turning successively about different centres. The analogue of the proposition for which the last figure was drawn is as follows:



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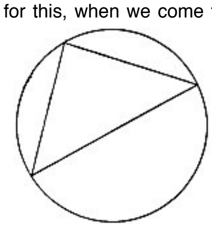
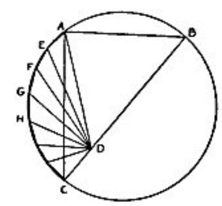
Upon a line, *a*, tangent to the absolute let two points be taken from which the other tangents to the absolute are *b* and *c*. Through the intersection of *b* and *c* draw any line, *d*, then any two lines *e* and *f*, meeting at the intersection of *a* and *b*, make the same angle with one another as two other lines having the same intersections with *d*, and cutting one another at the intersection of *a* and *c*. This enables us to compare all angles.

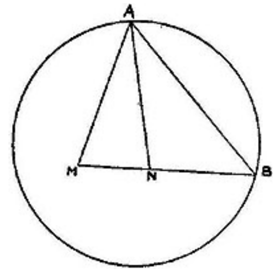
149. Suppose a man to be standing upon an infinitely extended plane free from all obstructions. Would he see something like a horizon line, separating earth from sky, being the foreshortened parts of the plane at an infinite distance? If space is infinite, he would. Now suppose he sets up a plane of glass and traces upon it the projection of that horizon, from his eye as a centre. Would that projection be a straight line? Euclid virtually says, "Yes." Modern geometers say it is a question to be decided experimentally. As a logician, I say that no matter how near straight the line may seem, the presumption is that sufficiently accurate observation would show it was a conic section. We shall see the reason for this, when we come to study probable inference.



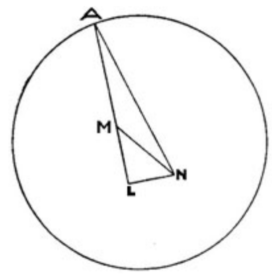
Let us suppose, then, that the horizon is not a straight line upon a level with the eye, but is a small circle below that level. If then two straight lines meet at infinity, their other ends must be infinitely distant; but the angle between them is null. Hence, there may be a triangle having all its angles null, and all its sides infinite. Let us assume (what might, however, be proved) that two triangles, having all the sides and angles of the one respectively and in their order equal to those of the other, are of equal area. Then all triangles having the sum of their angles *null* are of equal area. Call this *T*. Then the area of an ordinary polygon of *V* vertices all on the absolute is $(V\text{-}2)T$. The area of the absolute is therefore infinite.



If a triangle has two angles null and the third $1/N$ part of 180°, what is its area? Let *ABD* be the triangle, *AB* being on the absolute. Continue *BD* the absolute at *C*. Let *ADE*, *EDF*, etc., be N-1 triangles having their angles at *D* all equal to *ADB*. Then these *N* triangles are all equal, because their sides and angles are equal. They make a polygon of N+1 vertices on the absolute, the area of which is $(N\text{-}1)T$. Hence, the area of each triangle is $(1\text{-}1/N)T$.

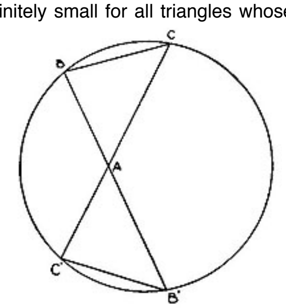


What is the area of a triangle having one angle zero and the others mX180° and nX180°? Let *AMN* be such a triangle; extend *MN* on the side of *N* to the absolute at *B*. Then the area of *ABM* is $(1-m)T$ and that of *ABN* is nT. Hence the area of *AMN*, which is their difference, is $(1-m-n)T$.



What is the area of a triangle having its three angles equal to lX180°, mX180°, and nX180°? Let *LMN* be such a triangle. Produce *LM* on the side of *M* to the absolute at *A* and join *AN*. Then if the angle *MNA* is put equal to x.180°, the area of *ALN* is $(1-l-n-x)T$, while that of *AMN* is $(m-x)T$. Hence, that of *LMN*, which is their difference, is $(1-l-m-n)T$.

Thus, the area of a triangle is proportional to the amount by which the sum of its angles falls short of two right angles. Of course, this does not forbid that amount being infinitely small for all triangles whose sides are finite.



150. The above reasoning may appear to be fallacious because it forgets that subtraction is not applicable to infinites. But it does not fall into that error. I may remark, however, that subtraction is applicable to infinites, in case their transformations are so limited that $x+y$ cannot equal $x+z$ unless $y = z$. For instance, we have considered the triangle *ABC* having two vertices on the absolute. This triangle is finite. But we might perfectly well reason about the infinite sector *ABC* provided this sector be not allowed to vary so as to change the area of the triangle, and provided, further, that we always add to each sector, *BAC*, its equal vertical sector *B'AC'*.

Looking at a triangle from this point of view, we see that the sum of the six sectors (two for each angle) is twice three times the triangle *plus* all the rest of the plane, or twice the area of the triangle plus the whole plane. The whole plane is four right angled sectors. But we have thus reckoned together with the sectors of the angles of the triangle their equal vertical sectors. Dividing by two, we find the sum of the angular sectors of a triangle is two right angles *plus* the area of the triangle.

Now since the sector is proportional to its angle, and since further, for the largest possible sector the angle is zero, it follows that the sector is equal to the negative of the angle, whence we find

$$\text{area } \Delta = 2 \perp - \Sigma \text{ Angles.}$$

151. . . . Such are the ideas which the mathematician is using every day. They are as logically unimpeachable as any in the world; but people, who are not sure of their logic, or who, like many men who pride themselves on their soundness of reason, are totally destitute of it, and who substitute for reasoning an associational rule of thumb, are naturally afraid of ideas that are unfamiliar, and which might lead them they know not whither.

As compared with imaginaries, with the absolute, and with other conceptions with which the mathematician works fearlessly — because good logicians, the Cauchys and the like,[†P1] have led the way — as compared with these, the idea of an infinitesimal is exceedingly natural and facile. Yet men are afraid of infinitesimals, and resort to the cumbrous method of limits. This timidity is a psychological phenomenon which history explains. But I will not occupy space with that here.[†1]

It was Fermat, a wonderful logical and still more wonderful mathematical genius, whose light was almost extinguished by the bread-and-butter difficulties which the secret plotting of worldlings forced upon him, who first taught men the method of reasoning which lies at the bottom of all modern science and modern wealth, the method of the differential calculus.[†2] He gave a variety of instructive examples, did this lawyer, this "conseiller de minimis," as the jealous Descartes was base enough to call him, joining himself to the "born missionaries" who were determined to "head off" this hope of mankind. But the first and simplest of them is the solution of the problem to divide a number, a, into two parts so that their product shall be a maximum. Let the parts be x and $a-x$. Let e be a quantity such that $a+e$ is "adequal" or {parisos}, say perequal to a. Then, the product being a maximum is at the point when increase of x ceases to cause it to increase. Hence Fermat writes

$$x(a-x) = (x+e)(a-x-e)$$

which gives

$$0 = -xe+e(a-x-e)$$

or

$$0 = e(a-2x-e)$$

Fermat now divides both sides by *e* (which assumes *e* is not zero). Whence

$$0 = a-2x-e.$$

But a-2x-e is "adequal" to a-2x; and the *e* may consequently be "elided." Thus we get

$$0 = a-2x,$$

or

$$x = (1/2)a.$$

The peculiar properties of *e*, which we now call, after Leibniz, the infinitesimal, are:

First, that if $pe = qe$, then $p = q$, contrary to the property of zero; while

Second, that, under certain circumstances, we treat *e* as if zero, writing

$$p+e = p.$$

Of course, we cannot adopt the last equation without reservation. For it would follow that

$$e = 0,$$

whence, since

$$4\times0 = 5\times0,$$
$$4e = 5e,$$

and then by the first property,

$$4 = 5.$$

The method of indivisibles [†P1] had recognized that infinitely large numbers may have definite ratios, so that division is applicable to them.

152. The simplest way of defending the algebraical device is to say that *e* represents a quantity immeasurably small, that is, so small that the Fermatian inference does not hold from these quantities to any that are assignable. That no contradiction is involved in this has been shown in the former part of this chapter.[†1] In the sense of **measurement** , then, $p+e = p$, while from a formally logical point of view, it is assumed that $e>0$. This is the most natural way, a perfectly logical way, and the way the most consonant with modern mathematics.

It is also possible to conceive the reasoning to represent the following. (The problem is the same as above.) Let *x* be the unknown. Then, since $x(a-x)$ is a maximum,

$$x(a-x)>(x+e)(a-x-e)$$

for all neighboring values of *e*. That is

$$0>e(a-2x-e).$$

Then the sign of $a-2x-e$ is opposite to that of *e* no matter what the value of *e*. It follows that 2x differs from *a* by less than any assignable quantity.

The great body of modern mathematicians repudiate infinitesimals in the above literal sense, because it is not clear that such quantities are possible, or because they cannot entirely satisfy themselves with that mode of reasoning. They therefore adopt the method of limits, which is a method of establishing the fundamental principles of the differential calculus. I have nothing against it, except its timidity or inability to see the logic of the simpler way. Let *x* be a variable quantity which takes an unlimited series of values x_1, x_2, . . . x_n, so that *n* will be a variable upon which x_n, depends. If, then, there be a quantity *c* such that

$$x_\infty = c,$$

that is, as the mathematicians prefer to say, in order to avoid speaking of infinity, if for every positive quantity *e* sufficiently small, there be a positive quantity ν such that for all values of *n* greater than ν

$$\text{Modulus } (x_n-c)<e$$

then *c* is said to be the *limit* of *x*.

Upon this definition is raised quite an imposing theory about limits which I can only regard with admiration, when it is erected with modern accuracy. Only, I wish to point out that the need for such a definition is not limited to its application to $n = \infty$, nor because infinity presents peculiar difficulties. It is only because ∞ is not an assignable number with which we can perform arithmetical processes. Let the function $x_n = n^2$, then the same difficulty arises when $n = \Pi$, and the same definition of a limit is called for.

The differential calculus deals with continuity, and in some shape or other, it is necessary to define continuity. I accept the above definition, with unimportant modifications, as a good definition of continuity. From it, as it appears to me, the idea of infinitesimals follows as a consequence; but, if not, no matter — so long as the algebraic expression of the infinitesimal be accepted, which is really the essential point. Infinitesimals may exist and be highly important for philosophy, as I believe they are. But I quite admit that as far as the calculus goes, we only want them to reason with, and if they be admitted into our reasoning apparatus (which is the algebra) that is all we need care for.